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Bending-Torsion Divergence of a Clamped–Clamped Composite Wing

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I. Introduction

THE analysis of the divergence of a fixed-free composite wing (with sweep) is considered by Hodges and Pierce [1] and has a known exact solution developed by Weisshaar [2] based on a generalization of the solution for swept isotropic wings given by Diederich and Budiansky [3]. In all these works, the aerodynamic theory is an inviscid, incompressible, two-dimensional strip theory, and the wing structure is modeled as a beam. Using the same type of model, we here consider the case of a spanwise-uniform, unswept, composite wing clamped at both ends. This case has not been treated in the literature to the best of the author's knowledge. Such a solution would apply, for example, to a wind-tunnel model clamped at both ends or to a lifting surface that connects the two fuselages of a dual-fuselage aircraft.

II. Analysis

Consider an elastically coupled wing, the strain energy of which is

$$U = \frac{1}{2}\int_0^\ell (\overline{EI}w''^2 - 2Kw''\theta' + \overline{GJ}\theta'^2)\,\mathrm{d}y \tag{1}$$

where w is the transverse deflection, θ is the elastic twist angle, $\overline{EI}$ is the bending stiffness, $\overline{GJ}$ is the torsional stiffness, K is the bending-twist elastic coupling stiffness, y is the axial coordinate along the wing reference line, and ℓ is the length. For isotropic wings, the bending stiffness $\overline{EI}$ would be the Young's modulus E times the sectional area moment of inertia I, and the torsional stiffness $\overline{GJ}$ would be the shear modulus G times the torsional constant J. However, for a composite wing, these stiffnesses along with K are complicated functions of the wing sectional geometry and the material properties. In any event, physics dictates that $K^2 < \overline{EI}\,\overline{GJ}$, but practicality dictates a more restrictive condition, that K^2 not exceed approximately $\overline{EI}\,\overline{GJ}/2$. Note that K can be positive, negative, or zero.

The wing loading is defined in terms of the distributed lift and its moment about the reference line, so that the virtual work is

$$\overline{\delta W} = \int_0^\ell qca_0\theta(\delta w + e\delta\theta)\,\mathrm{d}y \tag{2}$$

where q is the dynamic pressure, c is the wing chord, a_0 is the lift-curve slope, and e is the offset between the reference line and the aerodynamic center, positive when the latter is toward the leading edge. All these quantities are assumed to be constant along the wing.

For a spanwise-uniform, composite beam with bending-twist coupling, one cannot define an axis through which transverse shear forces act without twisting the beam as the locus of a cross-sectional property. Instead, the y axis must be the locus of generalized shear centers [4]. The generalized shear center is the point in the cross section at which transverse shear forces are structurally decoupled from the twisting moment. Although transverse shear forces acting at the y axis do not *directly* induce twist, the bending moment induced by the shear force will still induce twist when $K \neq 0$.

The weak form of the governing equilibrium equations can be found by setting $\delta U = \overline{\delta W}$, so that

$$\begin{aligned}&\int_0^\ell [(\overline{EI}w'' - K\theta')\delta w'' - qca_0\theta\delta w]\,\mathrm{d}y = 0\\ &\int_0^\ell [(\overline{GJ}\theta' - Kw'')\delta\theta' - qeca_0\theta\delta\theta]\,\mathrm{d}y = 0\end{aligned} \tag{3}$$

The two governing ordinary differential equations then become

$$\overline{EI}w'''' - K\theta''' - qca_0\theta = 0 \qquad \overline{GJ}\theta'' - Kw''' + qeca_0\theta = 0 \tag{4}$$

These two equations can be easily combined into one third-order equation for the twist by solving the first for w'''' and substituting the result into the second equation differentiated with respect to y. The resulting third-order equation is

$$\overline{GJ}\theta''' - \frac{K}{\overline{EI}}(K\theta''' + qca_0\theta) + qeca_0\theta' = 0 \tag{5}$$

or

$$\left(1 - \frac{K^2}{\overline{EI}\,\overline{GJ}}\right)\theta''' + \frac{qeca_0}{\overline{GJ}}\theta' - \frac{Kqca_0}{\overline{EI}\,\overline{GJ}}\theta = 0 \tag{6}$$

This third-order equation requires three boundary conditions, but only two are obvious in the clamped–clamped case, namely that $\theta(0) = \theta(\ell) = 0$.

III. Solution

The case of a fixed-free, swept, composite wing is considered by Hodges and Pierce [1] and has a known exact solution developed by Weisshaar [2] based on a generalization of the solution for swept isotropic wings given by Diederich [3]. For the special case of zero sweep angle, the third boundary condition is applied at the free end (say at $y = \ell$) and is given by

$$\left(1 - \frac{K^2}{\overline{EI}\,\overline{GJ}}\right)\theta''(\ell) + \frac{qeca_0}{\overline{GJ}}\theta(\ell) = 0 \tag{7}$$

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The case of the clamped–clamped boundary conditions has not been treated in the literature to the best of the author's knowledge. Here we consider the case of a uniform wing with clamped–clamped boundary conditions, in which $\theta(0) = \theta(\ell) = w(0) = w'(0) = w(\ell) = w'(\ell) = 0$.

The governing equations in either Eqs. (4) or Eq. (6) can be simplified by letting $(\,)'$ be the derivative with respect to $\eta = y/\ell$ and introducing nondimensional parameters

$$\lambda^2 = \frac{qeca_0\ell^2}{\overline{GJ}} \qquad \kappa = \frac{K}{\sqrt{\overline{GJ}\,\overline{EI}}} \qquad A^2 = \frac{\overline{GJ}}{\overline{EI}} \qquad r = \frac{e}{\ell} \tag{8}$$

so that Eqs. (4) become

$$\theta'' - \frac{\kappa}{A}w''' + \lambda^2\theta = 0 \qquad \frac{r}{A^2}w'''' - \frac{r\kappa}{A}\theta''' - \lambda^2\theta = 0 \tag{9}$$

and Eq. (6) becomes

$$(1-\kappa^2)\theta''' + \lambda^2\theta' - \frac{\lambda^2}{z}\theta = 0 \tag{10}$$

where

$$z = \frac{r}{A\kappa} \tag{11}$$

The obvious boundary conditions become $\theta(0) = \theta(1) = 0$. To find a third boundary condition, however, is problematic. An expression for the third boundary condition may be found by following these steps:

1) Add Eqs. (9), obtaining a perfect differential:

$$\theta'' - \frac{\kappa}{A}w''' + \frac{r\kappa}{A}\theta''' - \frac{r}{A^2}w'''' = 0 \tag{12}$$

2) Integrate the perfect differential twice, resulting in a first integral with one arbitrary constant and a second integral with two:

$$\begin{aligned}\theta' - \frac{\kappa}{A}w'' + \frac{r\kappa}{A}\theta'' - \frac{r}{A^2}w''' &= c_1\\ \theta - \frac{\kappa}{A}w' + \frac{r\kappa}{A}\theta' - \frac{r}{A^2}w'' &= c_1\eta + c_2\end{aligned} \tag{13}$$

3) Solve the first of Eqs. (9) for w''', yielding

$$w''' = \frac{A}{\kappa}(\theta'' + \lambda^2\theta) \tag{14}$$

and solve the second of Eqs. (13) for w'', giving

$$w'' = \frac{A^2}{r}\left(\theta - \frac{\kappa}{A}w' + \frac{r\kappa}{A}\theta' - c_1\eta - c_2\right) \tag{15}$$

Using these expressions in the first of Eqs. (13), one obtains

$$\theta' - \frac{\kappa[A(xc_1 + c_2 - \theta) + \kappa(w' + r\theta')]}{r} - \frac{r\kappa\theta''}{A} + \frac{r(\theta\lambda^2 + \theta'')}{A\kappa} = c_1 \tag{16}$$

4) Using the fact that $w'(0) = w'(1) = 0$, we may now solve for c_1 and c_2 by evaluating Eq. (16) at $\eta = 0$ and $\eta = 1$.

5) Integrate Eq. (10), using the fact that $\theta(0) = \theta(1) = 0$ to find

$$\int_0^1 \theta\,\mathrm{d}\eta = \frac{z(1-\kappa^2)}{\lambda^2}[\theta''(1) - \theta''(0)] \tag{17}$$

6) Finally, we integrate Eq. (16) from $\eta = 0$ to $\eta = 1$, using Eq. (17), the fact that $w(0) = w(1) = 0$, and the values of c_1 and c_2, and we then obtain the final result for the third boundary condition as

$$\begin{aligned}&\theta'(0) + \theta'(1) + 2z[\theta'(0) - \theta'(1)] + z[\theta''(0) + \theta''(1)]\\ &\quad + 2\left(\frac{1}{\lambda^2} + z^2\right)[\theta''(0) - \theta''(1)] = 0\end{aligned} \tag{18}$$

Equation (10), along with the boundary conditions $\theta(0) = \theta(1) = 0$, plus Eq. (18), can be solved exactly, using such tools as Mathematica or Maple. The exact solution is of the form

$$\theta = K_1 e^{\beta_1\eta} + K_2 e^{\beta_2\eta} + K_3 e^{\beta_3\eta} \tag{19}$$

where the β_i for $i = 1$, 2, and 3, are the three roots of the cubic polynomial

$$z[\beta^3(1-\kappa^2) + \beta\lambda^2] - \lambda^2 = 0 \tag{20}$$

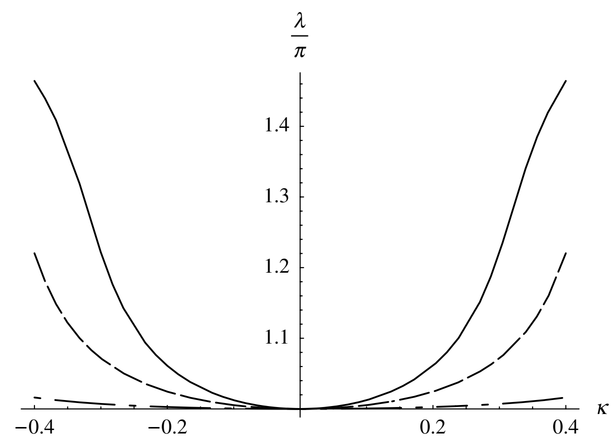

Fig. 1 Behavior of λ/π vs κ for $r/A = 0.0625$ (dot-dash line), $r/A = 0.0225$ (dashed line), and $r/A = 0.015$ (solid line).

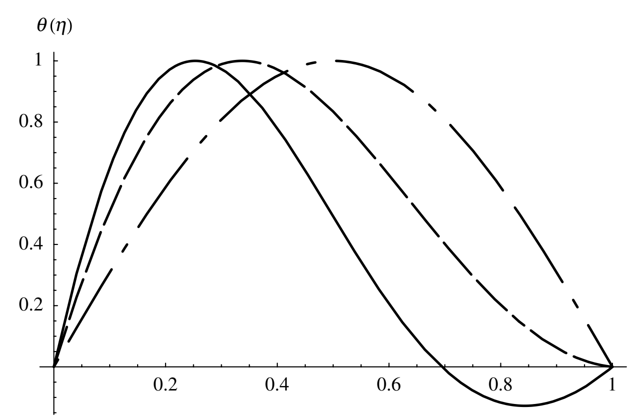

Fig. 2 Divergence mode shapes for θ with $r/A = 0.05$ and $\kappa = 0$ (dot-dash line), $\kappa = 0.2$ (dashed line), and $\kappa = 0.4$ (solid line).

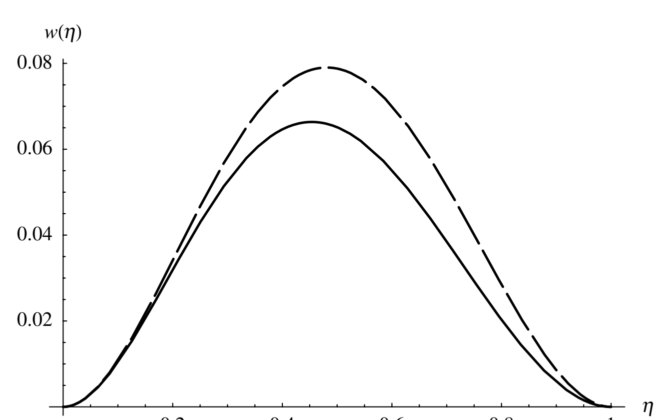

Fig. 3 Divergence mode shapes for w with $A = 0.2$, $r = 0.01$, and $\kappa = 0$ (identically zero), $\kappa = 0.2$ (dashed line), and $\kappa = 0.4$ (solid line).

Three homogeneous equations are formed by the three boundary conditions, and the coefficient matrix multiplying the three constants K_i must have a determinant equal to zero for a nontrivial solution to exist. The value of λ is unknown and plays the role of an eigenvalue with the divergence dynamic pressure given by $q_{\mathrm{div}} = \overline{GJ}\lambda^2/(eca_0\ell^2)$.

When $\kappa = 0$, the result of $\lambda = \pi$ is obtained. The value of λ does not depend on the sign of κ, because the wing with a $\kappa = \kappa^* > 0$ is identical to a mirror image about the midspan of one with $\kappa = -\kappa^*$ and hence has the same divergence dynamic pressure (though with the mirror-image mode shape). The exact solution of Eq. (10) for λ depends only on $|\kappa|$ and the ratio r/A, and is plotted vs κ in Fig. 1 for specific values of r/A. The value of λ rises as $|\kappa|$ increases, and for smaller r/A the sensitivity increases.

The divergence mode shapes include both θ and w. Knowing c_1 and c_2, the latter can be obtained by integration of Eq. (16) to find

$$\begin{aligned}\frac{2\kappa w(\eta)}{A} &= 2(z^2\lambda^2 + 1)\int_0^\eta \theta(\zeta)d\zeta \\ &+ z(1-\kappa^2)\langle 2\theta + [(\eta-2)\eta - 2z]\theta'(0) - \eta^2\theta'(1) \\ &+ z\{2\theta' + \eta[(\eta-2)\theta''(0) - \eta\theta''(1)]\}\rangle \end{aligned} \tag{21}$$

The solution for the mode shapes, unlike that for λ, is clearly dependent on κ, r, and A. At $\kappa = 0$, $w \equiv 0$, and $\theta = \sin(\pi\eta)$ is symmetric about the midspan. Both w and θ become nonsymmetric for nonzero κ and more so as $z = r/(A\kappa)$ decreases (i.e., as either κ or the sensitivity of λ to κ increases), as shown in Figs. 2 and 3 for $\kappa \geq 0$. Note that mode shapes are normalized so that the maximum value of θ is unity. For $\kappa < 0$, the mode shapes are mirror images about the midspan of their $\kappa > 0$ counterparts.

Obviously, exact solutions of Eqs. (9) and their boundary conditions are identical to the solution presented herein. Moreover, results from the Ritz method using polynomial admissible functions for w and θ (not presented herein), applied to Eqs. (3), agree well with the exact solution. Finally, unlike the much simpler clamped–free case [4], attempts here to apply an assumed mode method to the third-order equation, Eqs. (10), for the clamped–clamped case were not successful.

References

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